On the geography of threefolds of general type with small volume

joint work with S. Coughlan [C], Y. Hu [H], T. Zhang [Z]

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Overview

1. Volume and genus

2. Fibrations

3. Threefolds on the Noether line

4. Moduli spaces

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Volume and genus: what they are

Let X be a complex projective variety whose singularities are at worst canonical. Let n be the dimension of X.

Definition

The genus of X is $p_g(X) := h^0(X, K_X)$.
The volume of X is

$$\text{vol}(X) := n! \limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^n}$$

A variety is of general type if and only if its volume is positive.
X is a canonical model if the singularities of X are canonical and K_X is ample.

Both genus and volume are birational invariants.
If X is a canonical model then

$$\text{vol}(X) = K_X^n.$$

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Volume and genus: Noether inequality

If X is a curve ($n = 1$) then $2\mathbb{N} \ni \text{vol}(X) = 2p_g(X) - 2$.
If X is a surface ($n = 2$) we have the Noether Inequality

$$\mathbb{N} \ni \text{vol}(X) \geq 2p_g(X) - 4$$

If X is a threefold ($n = 3$) we have now the Noether inequality

Theorem (Chen, J. A.; Chen, M.; Jiang, C., Kollár, J., H)

Let X be a canonical 3-fold. Then

$$\mathbb{Q} \ni \text{vol}(X) \geq \frac{4}{3}p_g - \frac{10}{3}$$

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Examples of Kobayashi in the 90s show that the inequality is sharp for
infinitely many p_g.
In fact we know more.

Volume and genus: Noether inequality

We have the refined Noether inequality

> **Theorem (Chen, J- A.; Chen, M.; Jiang, C., Kollár, J., H, Z)**
>
> *Let X be a canonical 3-fold. Then*
>
> $$\mathbb{Q} \ni \mathsf{vol}(X) \geq \frac{1}{2} \left\lceil 2 \left(\frac{4}{3} p_g - \frac{10}{3} \right) \right\rceil \in \frac{1}{2} \mathbb{N}$$

This is sharp for all $p_g \geq 3$.
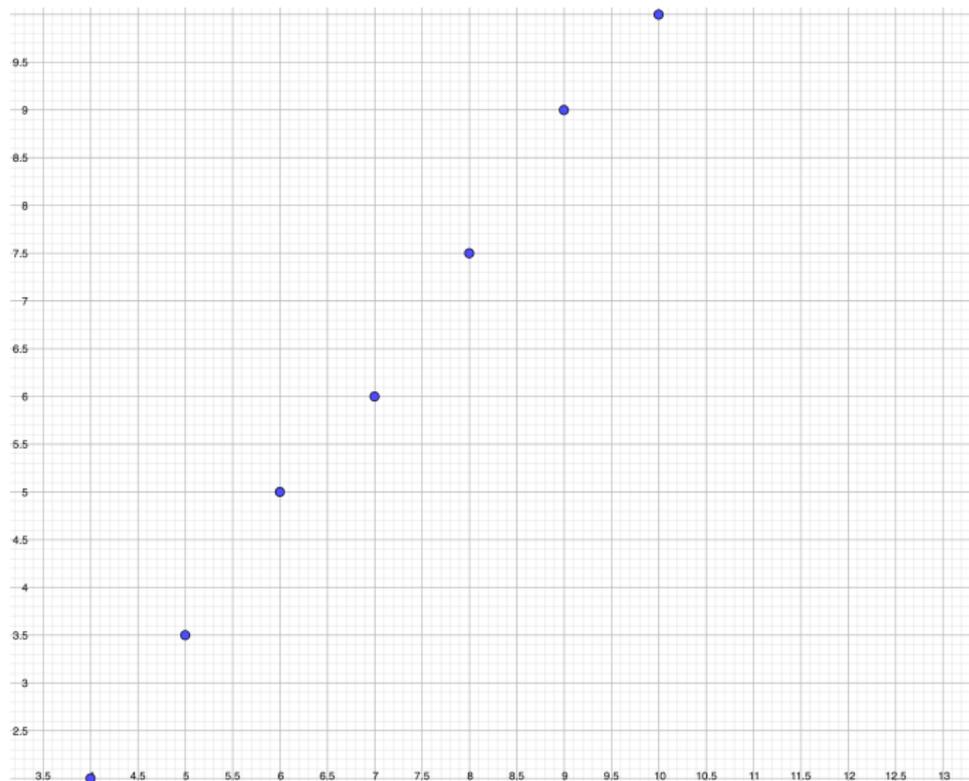
We say the a threefold X stays on the **refined** Noether line if $\mathsf{vol}(X) = \frac{1}{2} \left\lceil 2 \left(\frac{4}{3} p_g - \frac{10}{3} \right) \right\rceil$.
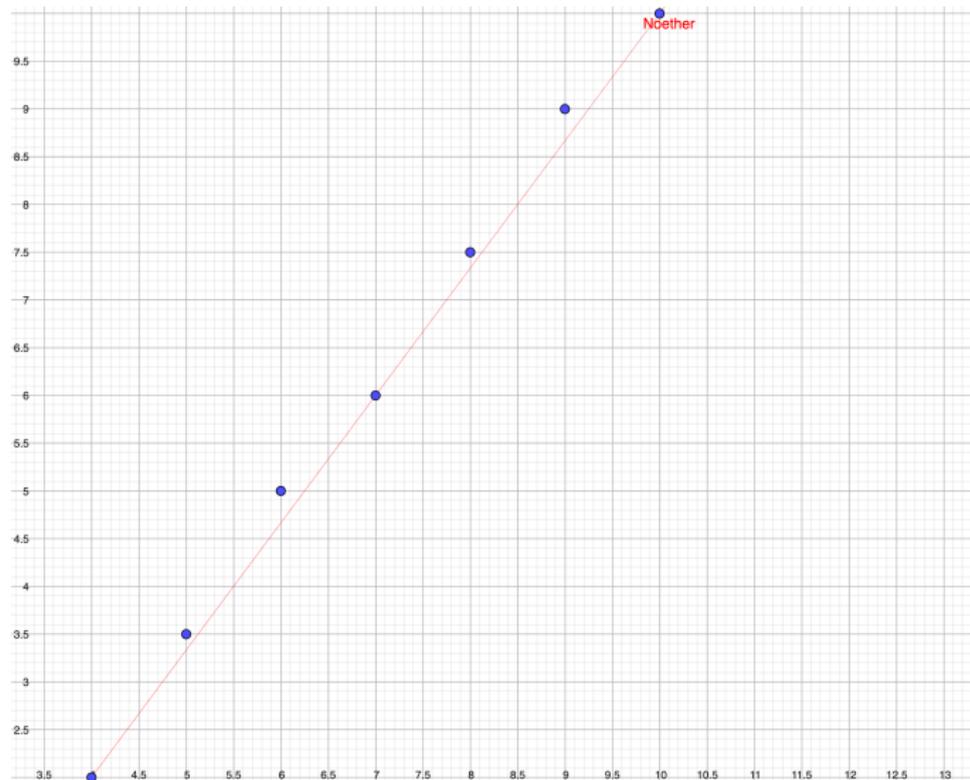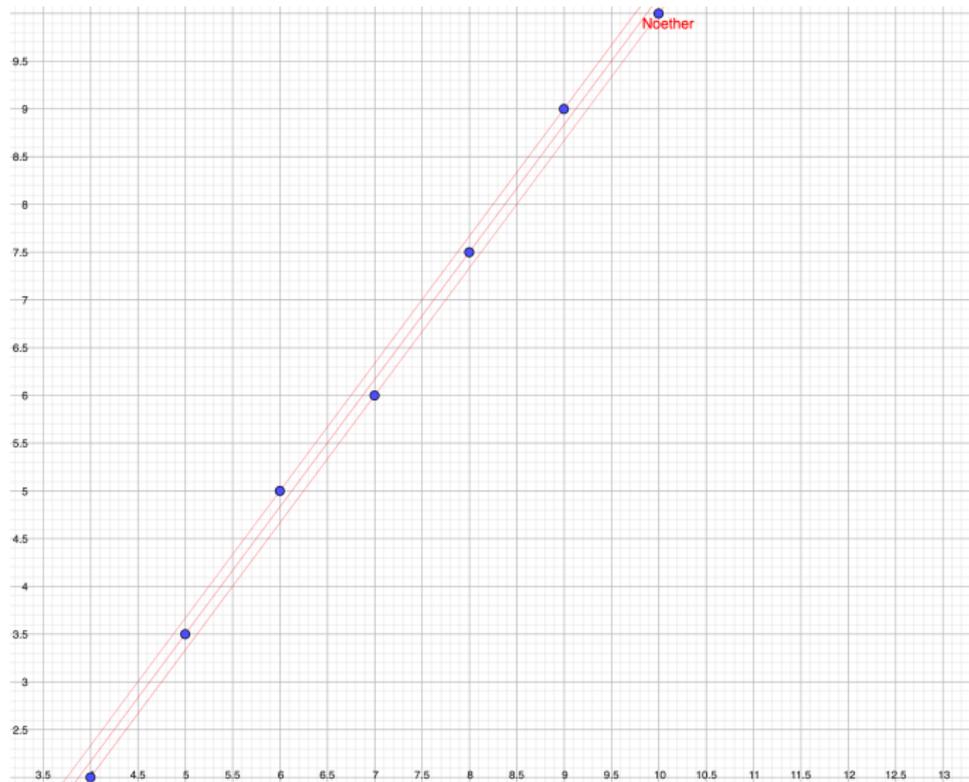Well, this is in fact not a line, but three...

The minimal volumes

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The minimal volumes and the Noether line

The minimal volumes and the refined Noether line

Surfaces on the Noether line

The canonical surfaces X with $K_X^2 = 2p_g(X) - 4$ are usually referred to as Horikawa surfaces because Horikawa described their moduli space.
We summarize here only his results for higher p_g, that are the most important for this talk.

Theorem (Horikawa, E. 1976)

Fix an integer $p_g \geq 7$.
All canonical surfaces X with geometric genus p_g and volume
$K_X^2 = 2p_g(X) - 4$ *have a fibration[a] $f \colon X \to \mathbb{P}^1$ in curves of genus 2.*
The moduli space of the canonical surfaces X with geometric genus p_g and volume $K_X^2 = 2p_g(X) - 4$

is unirational if K_X^2 is not divisibile by 8;

has two connected components, both unirational, if K_X^2 is divisibile by 8.

___
[a]surjective morphism with connected fibres

Genus 2 fibrations

Let $f \colon X \to B$ be a genus 2 fibration, that is a surjective morphism onto a projective curve B whose general fibre is a curve of genus 2. Let F be a fibre.

By the theory of genus 2 fibrations, the canonical ring of F, that is the ring $\bigoplus_d H^0(F, dK_F)$ is of one of the following two forms:

If F is 2−connected

$$\frac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)}$$

$$\deg x_j = 1 \qquad \deg z = 3$$
$$\deg f_6 = 6$$

If $F = C_1 + C_2$ with $C_1 C_2 = 1$

$$\frac{\mathbb{C}[x_0, x_1, y, z]}{f_2(x_0, x_1), f_6(x_0, x_1, y, z)}$$

$$\deg x_j = 1 \quad \deg y = 2 \quad \deg z = 3$$
$$\deg f_2 = 2 \quad \deg f_6 = 6$$

Invariants of genus 2 fibrations

If X is moreover a canonical surface, then[1]

$$\mathrm{vol}(X) = 2p_g(X) - 4 + 6b - 2h^1(X, \mathcal{O}_X) + \deg \tau$$

where b is the genus of the base curve B, and τ is an effective divisor on B supported on the image of the *2-disconnected* fibres, those of the form $C_1 + C_2$ with $C_1 C_2 = 1$.

By an inequality of Jongmans and Debarre, if $K_X^2 < 2p_g$, then $h^1(X, \mathcal{O}_X) = 0$, which in turn implies $b = 0$.

In particular

$$K_X^2 = 2p_g(X) - 4 \Leftrightarrow h^1(X, \mathcal{O}_X) = 0 \text{ and } \tau = 0.$$

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[1]Catanese, F., Pignatelli, R.; *Fibrations of low genus, I.* Annales Scientifiques de l'École Normale Supérieure, **39**, 6 (2006), 1011–1049

Simple genus 2 fibrations

We can reformulate more precisely the previous remark as follows:

Definition

A simple genus 2 fibration is a morphism $f : X \to B$ between projective varieties of respective dimension 2 and 1 such that

1. B is smooth;
2. all singularities of X are canonical;
3. K_X is f-ample;
4. for *all* $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form $\frac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)}$, with $\deg x_i = 1$, $\deg z = 3$, $\deg f_6 = 6$

Theorem

Let X be a canonical surface with $p_g \geq 7$. Then $K_X^2 = 2p_g - 4$ if and only if X is a regular[a] simple genus 2 fibration.

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[a]regular means $h^1(X, \mathcal{O}_X) = 0$, which implies $b = 0$.

Fibrations in (1,2)-surfaces

The first step of the proof of the Noether inequality for threefolds is the proof that

$$\text{vol}(X) < 2p_g(X) - 6 \Rightarrow \text{ there exists a fibration } f\colon X \dashrightarrow \mathbb{P}^1$$

such that the general fibre of f is a $(1,2)$-surface.

Definition

A $(1,2)$-surface is a canonical surface[a] with $K^2 = 1$ and $p_g = 2$.
They are all hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.

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[a]canonical model of dimension 2

A classification of the canonical rings of all possible degenerations of families of $(1,2)$-surfaces seems out of reach.

However, the previous analysis suggests the following definition

Definition (CP 2023[a])

[a]Coughlan, S.; Pignatelli, R.; *Simple fibrations in (1,2)-surfaces*, Forum of Mathematics, Sigma, Volume **11**, 2023, e43

A simple fibration in $(1,2)$-surfaces is a morphism $f \colon X \to B$ between projective varieties of respective dimension 3 and 1 such that

1. B is smooth;
2. all singularities of X are canonical;
3. K_X is f-ample;
4. for *all* $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form[a]

$$\frac{\mathbb{C}[x_0, x_1, y, z]}{f_{10}(x_0, x_1, y, z)}$$

[a]$\deg x_i = 1$, $\deg y = 2$, $\deg z = 5$, $\deg f_{10} = 10$

The main theorems

Theorem (CHPZ 2025)

Suppose that X_{can} is a canonical threefold on the refined Noether line with $p_g \geq 5$.
Then X_{can} is regular and there is a crepant birational morphism[a] $X \to X_{can}$ and a simple fibration in $(1,2)$-surfaces $f: X \to \mathbb{P}^1$.
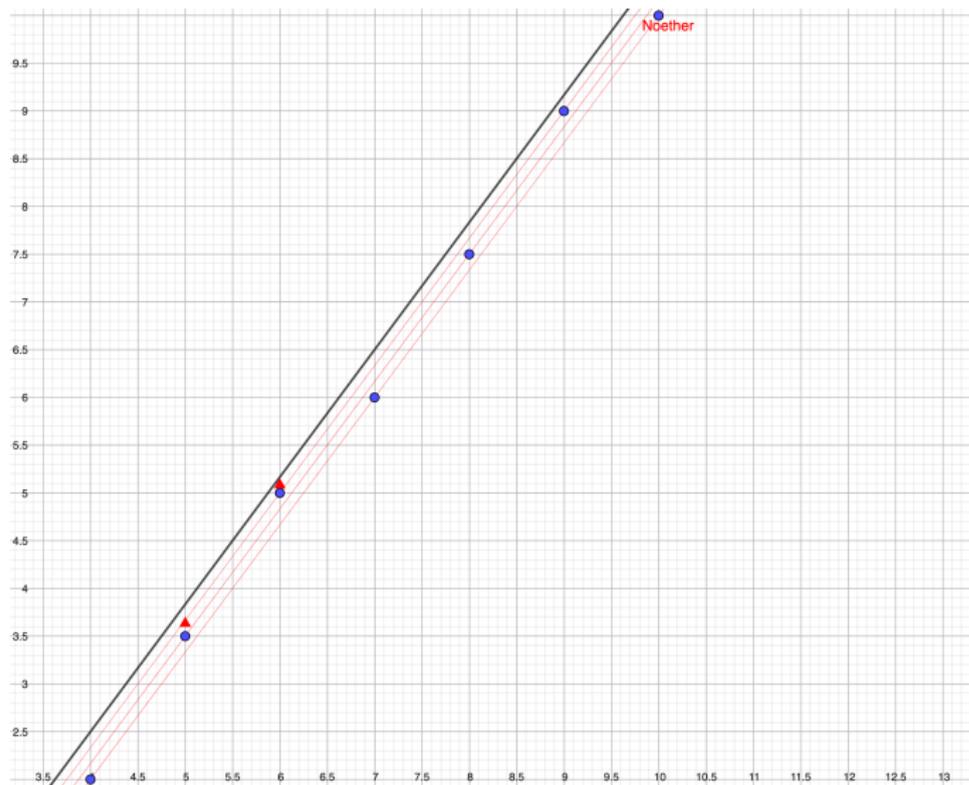*If $p_g \geq 23$, then $X_{can} \cong X$.*

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[a]A small contraction

In fact the proof shows also that if X_{can} is a canonical threefold with

- either $p_g \geq 7$ and $vol(X) < \frac{4}{3}p_g - \frac{10}{3} + \frac{1}{2}$
- or $p_g = 6$ and $vol(X) < \frac{61}{12} = \frac{4}{3}p_g - \frac{10}{3} + \frac{5}{12}$
- or $p_g = 5$ and $vol(X) < \frac{109}{30} = \frac{4}{3}p_g - \frac{10}{3} + \frac{3}{10}$

then X_{can} is on the refined Noether line.

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The minimal volumes and the refined Noether line

Theorem (CP 2023+HZ 2025[a])

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[a]This is a special case of Proposition 4.21 of [CP 2023] modified by using $h^2(X, \mathcal{O}_X) = 0$, proved in [HZ 2025]

Let $f \colon X \to \mathbb{P}^1$ be a simple fibration in $(1,2)$-surfaces. Then X is 2-Gorenstein.
Moreover

$$K_X^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{N}{6}$$

where N is the number of the points of X that are not Gorenstein, counted with multiplicity.

These two results tell us that we can classify the canonical threefolds on the refined Noether line, for $p_g \geq 5$, by classifying simple fibration in $(1,2)$-surfaces with $N \leq 2$, which, as we will see, is much simpler.

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$\mathbb{P}(1, 1, 2, 5)$-bundles

Every simple fibrations in $(1, 2)$-surfaces is naturally embedded as a divisor in a 4-fold $\mathbb{F} \to B$ that is a bundle[2] in weighted projective spaces $\mathbb{P}(1, 1, 2, 5)$, a divisor of relative degree 10.

The following is a way to construct toric 4-folds $\mathbb{F} \to \mathbb{P}^1$ that are bundles in weighted projective spaces $\mathbb{P}(1, 1, 2, 5)$: consider $\mathbb{C}^6$ with coordinates t_0, t_1, x_0, x_1, y, z.

Take a toric 4-fold $\mathbb{F} = \mathbb{C}^6 // (\mathbb{C}^*)^2$ defined by the weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & a & b & c & d \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$. Then (t_0, t_1) defines a bundle $f : \mathbb{F} \to \mathbb{P}^1$ in weighted projective spaces $\mathbb{P}(1, 1, 2, 5)$.

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[2]Here the word "bundle" means that all fibres are isomorphic to $\mathbb{P}(1, 1, 2, 5)$

Divisors in $\mathbb{P}(1,1,2,5)$-bundles

Theorem

Let $f: X \to \mathbb{P}^1$ be a simple fibration in $(1,2)$-surfaces with[a] $N \leq 4$. Then $X =: X(d; d_0, N)$ is a divisor in the toric 4-fold $\mathbb{F} = \mathbb{F}(d; d_0, N)$ given by the weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d - d_0 & d_0 - 2d - N & -N & -2N \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$, given by a bihomogeneous equation of bidegree $(-4N, 10)$ of the form

$$z^2 = f_N(t_0, t_1) y^5 + \ldots$$

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[a]The result is sharp as we have examples of simple fibration in $(1,2)$-surfaces with $N = 5$ such that $\mathbb{F}$ is not toric.
Note that to be on the refined Noether line correspond to $N \leq 2$.

$X(d; d_0, N)$

Fix $d, d_0 \in \mathbb{Z}$, $N \in \{0, 1, 2\}$. Direct computations show:

1. The general $X(d; d_0, N)$ is a simple fibration in $(1, 2)$-surfaces[3] iff

$$\frac{1}{4}(d + N) \leq d_0 \leq \frac{1}{2}(3d + N)$$

2. If $\min(d, d_0) \geq 1$ then $p_g(X) = 3d - 2 + N$, $K_X^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{N}{6}$. In particular d and N are deformation invariant.

3. The image of the canonical map of $X(d; d_0, N)$ is a rational normal curve if $d_0 = 1$, the cone over a rational normal curve if $d_0 = 2$, the Hirzebruch surface $\mathbb{F}_{3d-2d_0+N}$ if $d_0 \geq 3$.

4. K_X is big and nef[4] iff $\min(d; d_0) \geq 2$ and ample iff $\min(d; d_0) \geq 3$.

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[3] i.e. it has only canonical singularities

[4] The cases with $\min(d; d_0) = 2$ are those where $X_{can} \neq X$. In these cases there is a curve, a section of $f : X \to \mathbb{P}^1$, orthogonal to K_X, contracted to X_{can}

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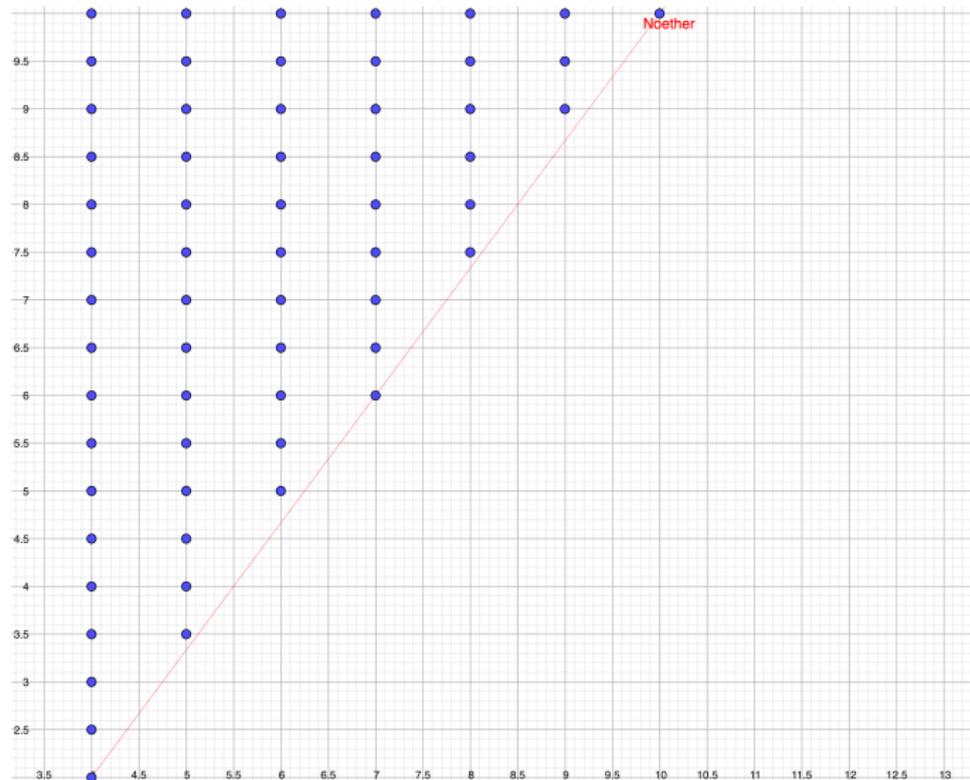
A components of the main stream

Theorem

Assume $p_g = 3d - 2 + N \geq 5$, $N \leq 2$.

If $d_0 \geq d + \frac{N}{2}$ then the threefolds $X(d; d_0, N)$ belongs all to the same irreducible component of the moduli space of canonical threefolds. The threefolds $X\left(d; \left\lceil \frac{1}{2}(3d + N) \right\rceil, N\right)$ form an open subset of it.

Theorem (CP2023)

The general element of $X\left(d; \left\lceil \frac{1}{2}(3d) \right\rceil, 0\right)$ is a Mori Dream Space with Cox ring

$$\mathbb{C}[t_0, t_1, x_0, x_1, y, z]/(z^2 - \ldots)$$

The other irreducible components

For each $d_0 < d + \frac{N}{2}$, $d_0 \geq \frac{1}{4}(d + N)$ we have a further unirational stratum of the moduli space of the canonical threefolds on the refined Noether line with $p_g = 3d - 2 + N$.

We have proven that almost all these stratum are further irreducible components of the moduli space, showing that the number of irreducible components, in contrast with the surface case, grows linearly with p_g.

For sake of simplicity we report here only our results for $p_g \geq 11$.

Theorem

For an integer $p_g \geq 11$, let M_{K^3, p_g} be the coarse moduli space parameterizing all canonical threefolds on the refined Noether line with geometric genus p_g. Let $N \in \{0, 1, 2\}$ such that $N \equiv p_g + 2 \pmod 3$. Then

(1) M_{K^3, p_g} is a union of α_{p_g} unirational strata, where

$$\alpha_{p_g} = \begin{cases} \left\lfloor \frac{p_g + 6}{4} \right\rfloor, & \text{if } N = 0, 2; \\ \left\lfloor \frac{p_g + 8}{4} \right\rfloor, & \text{if } N = 1. \end{cases}$$

(2) The number ν_{p_g} of irreducible components is at most α_{p_g} and at least $\alpha_{p_g} - \beta_{p_g}$, where the value of β_{p_g} is given in the following table.

	$N = 0$	$N = 1$	$N = 2$
β_{p_g}	$\left\lfloor \frac{p_g + 8}{78} \right\rfloor$	$\left\lfloor \frac{p_g + 61}{78} \right\rfloor$	$\left\lfloor \frac{p_g + 36}{78} \right\rfloor$

In particular, ν_{p_g} grows linearly with p_g, as $p_g / 4$.

An open problem

It follows the following natural
Question: *Find the maximal $\epsilon > 0$ (if it exists) such that if $p_g >> 1$ then all threefolds X with $\mathrm{vol}(X) < \frac{4}{3} p_g(X) - \frac{10}{3} + \epsilon$ admit (birationally?) a regular simple fibrations in $(1,2)-$surfaces $f \colon X \to \mathbb{P}^1$.*

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Our results shows

$$\epsilon \geq \frac{1}{2}$$

The next examples show that

$$\epsilon \leq \frac{2}{3}$$

Fibrations in $(1,2)$−surfaces that are not simple

Let $\mathbb{F}$ be a toric 5-fold with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & u & z \\ 1 & 1 & -a & -a & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, u, z)$. Two bihomogeneous equations of the form

$$z^2 + y^5 = \ldots \qquad\qquad t_0 u = \ldots$$

defines a **3-Gorenstein** threefold X with 1 non Gorestein point, at $t_0 = x_0 = x_1 = y = z = 0$, $t_1 = u = 1$, of type $\frac{1}{3}(1,2,2)$.
The map $X \to \mathbb{P}^1$ is a fibration in $(1,2)$-surfaces that is not simple[6].

$$p_g(X) = 6a \qquad\qquad K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{2}{3} \in \frac{1}{3}\mathbb{N} \setminus \mathbb{N}$$

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[6]because X is not 2−Gorenstein